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Arbitrage opportunities in diverse markets via a non-equivalent measure change

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Abstract We study arbitrage opportunities in diverse markets as introduced by Fernholz (J Math Econ 31:393–417, 1999). By a change of measure technique we are able to generate a variety of diverse markets. The construction is based on an absolutely continuous but non-equivalent measure change which implies the existence of instantaneous arbitrage opportunities in diverse markets. For this technique to work, we single out a crucial non-degeneracy condition. Moreover, we discuss the dynamics of the price process under the new measure as well as further applications.

Keywords Admissible strategies · Arbitrage opportunities · Diverse markets · Financial markets · Incomplete markets · Measure change · Optional decomposition theorem · Stochastic exponential

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1 Introduction

An interesting result of Fernholz in his inspiring work about stochastic portfolio theory, see the monograph (Fernholz 2002) for a detailed account, is the possibility of arbitrage in markets where no stock is ever allowed to dominate the entire

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market in terms of market capitalization. The associated notion of a diverse financial market was introduced and studied by Fernholz in the papers Fernholz (1999, 2001) and the monograph (Fernholz 2002). In particular, Fernholz et al. (2005) have shown that arbitrage opportunities relative to the market portfolio exist over any given time-horizon.

The requirement that financial markets are diverse seems to be reasonable from a regulatory point of view (otherwise we would encounter a very different society). If the market share of an existing company exceeds certain thresholds, there will be restrictions imposed on the company which try to prevent it from increasing its market share. One can also think about regulation authorities controlling mergers and acquisitions.

The purpose of this paper is to give a somewhat generic construction which yields a multitude of diverse markets. The existence of arbitrage opportunities follows then immediately by the very nature of this construction. This allows for a very transparent explanation of this phenomenon. Our main idea is as follows: we start with a non-diverse arbitrage-free market by specifying the dynamics of the price process under some local martingale measure P^0 . We then construct a diverse market by changing to another probability measure Q . Since Q is absolutely continuous but not equivalent to P^0 , a simple argument based on the optional decomposition theorem then yields the existence of an arbitrage opportunity. The main technical difficulty is to ensure that the market fulfils a certain non-degeneracy condition which makes the forementioned measure change work. We can show this for some standard models including that of Fernholz et al. (2005) by using a time change technique.

Furthermore, we study the dynamics of the price processes when seen under the new measure Q . We also include a brief discussion of existing approaches to the valuation of claims in case the model is complete with respect to P^0 . Finally, we show how a similar change of measure technique can also be employed in currency markets where some exchange rate mechanism has been superimposed.

Arbitrage opportunities in situations governed by an absolutely continuous but non-equivalent measure change have been studied in earlier works. Gossen-Dombrowsky (1992) (unpublished, we are grateful to H. Föllmer for providing us with this reference) studies a complete market model in which the price process is constrained to stay inside fixed boundaries. The construction of Delbaen and Schachermayer (1995) of arbitrage possibilities in Bessel processes is also based on a similar technique.

2 Arbitrage opportunities in diverse markets

2.1 Prelude

Here we introduce some kind of pre-model. Our main model of interest will later be obtained from this by an absolutely continuous but non-equivalent measure change. Let us first specify the (preliminary) dynamics of the price processes of n risky assets. Their dynamics are governed by a probability measure P^0 living on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P^0)$. The filtration $(\mathcal{F}_t)$ satisfies the usual assumptions of right-continuity and completeness with $\mathcal{F}_0$ being trivial.

Definition 2.1 The price process $X = (X_i)_{1 \leq i \leq n}$ is given as stochastic exponential $\mathcal{E}(M)$ of some n -dimensional continuous local P^0 -martingale M . We therefore have

$$\frac{dX_i(t)}{X_i(t)} = dM_i(t), \quad 1 \leq i \leq n, \quad t \geq 0.$$

The market so far is directly modelled under some martingale measure P^0 for X . In particular, this excludes arbitrage opportunities. The set of all probability measures equivalent to P^0 such that X is a local martingale will be denoted by $\mathcal{M}^e(X)$. As we do not necessarily assume that the market is complete, $\mathcal{M}^e(X)$ need not be a singleton. We assume that each company has one single share outstanding. Then we can define the relative market weights:

Notation 2.2 The relative market weight μ_i of the i th stock is given as

$$\mu_i(t) = X_i(t) / (X_1(t) + \cdots + X_n(t)).$$

The largest market weight is denoted by $\mu_{\max}(t) = \max_{1 \leq i \leq n} \mu_i(t)$.

The following notion of diversity was introduced by Fernholz (1999).

Definition 2.3 We fix a finite time horizon $T > 0$ and say that the market is *diverse* (up to time T) if there exists $\delta \in (0, 1)$ such that for every $i = 1, \dots, n$

$$\mu_i(t) < 1 - \delta, \quad \forall t \in [0, T] \quad P^0 - \text{a.s.}$$

We impose one additional condition on X (the ND stands for “non-degenerate”):

Assumption ND We have for some $T > 0$ and $\delta \in (0, 1)$ that

$$\begin{aligned} 0 &< \inf_{P \in \mathcal{M}^e(X)} P \left(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta \right), \\ 1 &> P^0 \left(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta \right). \end{aligned}$$

This implies in particular that the market is *not* diverse under P^0 . Later we shall give a sufficient condition for Assumption ND to hold and show that it is satisfied in the standard Itô model as studied in Fernholz et al. (2005).

2.2 Construction of diverse markets and the arbitrage opportunity

We now pass over to a diverse market, governed by a probability measure Q which we shall construct using a certain change of measure technique. Under this measure Q , we will be able to show arbitrage opportunities. For this we first have to define what we mean by arbitrage. Here we use the notion of arbitrage with respect to (general) admissible strategies as defined in Delbaen and Schachermayer (1995).

Definition 2.4 (Arbitrage opportunity) A predictable process H that is X -integrable for a semimartingale X is called admissible if $\int H \, dX$ is uniformly bounded from below. The semimartingale X satisfies the no-arbitrage property for admissible integrands under Q if H admissible and $\int_0^T H_t \, dX_t \geq 0$ Q -a.s. imply $\int_0^T H_t \, dX_t = 0$ Q -a.s.

We can now define our measure change which directly leads to the construction of diverse markets:

Definition 2.5 Assume ND. We define a probability measure Q absolutely continuous to P^0 via its Radon–Nikodym density

$$\frac{dQ}{dP^0} = \begin{cases} 0 & \text{if } \mu_{\max}(t) \geq 1 - \delta \text{ for some } t \in [0, T] \\ c & \text{else} \end{cases} \quad (1)$$

where c is a normalizing constant. Since

$$P^0 \left(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta \right) < 1$$

by ND, we can always find $c \in \mathbb{R}$ such that Q is a probability measure because dQ/dP^0 is not P^0 -a.s. equal to 0. As

$$P^0 \left(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta \right) > 0$$

by ND, Q is absolutely continuous with respect to P^0 but not equivalent. This is crucial for the existence of arbitrage opportunities.

Remark 2.6 The filtration $(\mathcal{F}_t)$ typically does not satisfy the usual conditions with respect to Q . However, we refer to the Remark following Theorem 1 in Delbaen and Schachermayer (1995) for a remedy: consider the filtration $(\mathcal{G}_t)$ obtained from $(\mathcal{F}_t)$ by adding all Q -null sets. The results in Delbaen and Schachermayer (1995) then show that whenever we have a stopping time τ_Q and a $(\mathcal{G}_t)$ -predictable process H_Q there exists a stopping time τ_P and a $(\mathcal{F}_t)$ -predictable process H_P such that Q -a.s. $\tau_Q = \tau_P$ and H_Q and H_P are Q -indistinguishable. This implies that whenever we need to work with a $(\mathcal{G}_t)$ -predictable process we can essentially replace it by an $(\mathcal{F}_t)$ -predictable process. We shall always do so without further notice.

The dynamics of X with respect to Q can be described via Lenglart's extension of Girsanov's theorem, which we will discuss in Section 3. Hence we get many examples of diverse markets by our construction: every price process X as above, satisfying ND, leads to a diverse market when seen under Q .

Let us now construct an arbitrage opportunity under Q via an admissible strategy by using an extension of the argument given in Gossen-Dombrowsky (1992) and Delbaen and Schachermayer (1995) and applying it to our setting of incomplete markets. We shall make use of the following optional decomposition theorem (see, Föllmer and Kramkov 1997).

Theorem 2.7 (Optional decomposition theorem) Consider a process V which is bounded from below and a P -supermartingale for all $P \in \mathcal{M}^e(X)$ (which is non-empty since $P^0 \in \mathcal{M}^e(X)$). Then there exists a predictable X -integrable process H and an increasing adapted process C with $C_0 = 0$ such that

$$V = V_0 + \int H \, dX - C.$$

We now state the main result about arbitrage opportunities:

Proposition 2.8 Consider a probability measure Q which is absolutely continuous but not equivalent to P^0 . If

$$\sup_{P \in \mathcal{M}^e(X)} P \left(\frac{dQ}{dP^0} > 0 \right) < 1 \quad (2)$$

then there exists an arbitrage opportunity under Q which can be realized via an admissible strategy.

Remark 2.9 Note that it follows from Assumption ND that Q as in Definition 2.5 is an absolutely continuous, non-equivalent probability measure (with respect to P^0) and fulfils Condition (2).

Proof We consider the claim

$$f := \mathbf{1}_{\left\{ \frac{dQ}{dP^0} > 0 \right\}}$$

and define the process $V = (V_t)_{t \geq 0}$ as

$$V_t = \operatorname{ess\,sup}_{P \in \mathcal{M}^e(X)} E_P(f \mid \mathcal{F}_t), \quad t \geq 0.$$

V is a P -supermartingale for all $P \in \mathcal{M}^e(X)$ (see, Föllmer and Kramkov (1997)). It follows that for $t \geq 0$,

$$V_t \geq E_{P^0}(f \mid \mathcal{F}_t) \geq 0 \quad P^0 - \text{a.s.}$$

By the optional decomposition theorem, there exist H and C as specified above such that

$$V = V_0 + \int H \, dX - C.$$

Moreover, H is admissible since $\int H \, dX$ is bounded from below: for $t \geq 0$,

$$\begin{aligned} \int_0^t H_s \, dX_s &= V_t - V_0 + C_t \geq -V_0 \\ &= - \sup_{P \in \mathcal{M}^e(X)} E_P(f) \geq -1 \quad P^0 \text{ and } Q - \text{a.s.} \end{aligned}$$

For H to be an arbitrage opportunity under Q , we need to check whether

$$\int_0^T H_s dX_s = f - V_0 + C_T > 0 \quad Q - \text{a.s.}$$

Since we have that $f = 1$ Q -a.s., $C_0 = 0$ and C is increasing, this holds in particular if $1 - V_0 > 0$ or

$$\sup_{P \in \mathcal{M}^e(X)} E_P(f) < 1,$$

which is equivalent to

$$\sup_{P \in \mathcal{M}^e(X)} P\left(\frac{dQ}{dP^0} > 0\right) < 1,$$

which is our Assumption (2). □

Remark 2.10 Here we have constructed an arbitrage opportunity with respect to admissible integrands in the sense of Delbaen and Schachermayer (1995). This means that the value of the arbitrage portfolio is bounded from below in absolute terms. In relative arbitrage, as in Fernholz (2002) or Fernholz et al. (2005), the arbitrage portfolio is bounded from below relative to the market portfolio, or perhaps some other well-defined portfolio. This amounts to a change in numeraire for the lower bound from (constant) riskless asset to market portfolio. In general, relative arbitrage is not the type of arbitrage we consider here, since the numeraire portfolio is not necessarily bounded, so the result here does not follow from Fernholz (2002) or Fernholz et al. (2005). Moreover, although the sum of our arbitrage portfolio and the market portfolio dominates the market portfolio, it is not necessarily bounded from below relative to the market portfolio as numeraire. We would like to thank the anonymous referee for pointing this out to us.

2.3 On the non-degeneracy condition

We now give a condition which guarantees ND for small enough time horizons T . This condition furthermore implies Condition (2). We recall that M is assumed to be a continuous local P^0 -martingale.

Theorem 2.11 *Assume $\mu_{\max}(0) < 1 - \delta$ and that there exists $0 < \varepsilon < \kappa$ such that for all $\mathbb{R}^n$ -valued processes η we have*

$$\varepsilon \int_0^t \|\eta(s)\|^2 ds \leq \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \int_0^t \eta_j(s) d[M_j, M_k]_s \eta_k(s) \leq \kappa \int_0^t \|\eta(s)\|^2 ds, \quad (3)$$

where $\|\cdot\|$ denotes the Euclidean norm. Then ND is satisfied for some $T > 0$ small enough where the probability space might possibly have been extended to support an independent Brownian motion.

Proof Let $T > 0$ to be chosen later. Fix $t \in [0, T]$. We have

$$\begin{aligned} d \left(\sum_{j=1}^n X_j(t) \right) &= \left(\sum_{j=1}^n X_j(t) \right) \sum_{k=1}^n \frac{dX_k(t)}{\sum_{j=1}^n X_j(t)} \\ &= \left(\sum_{j=1}^n X_j(t) \right) \sum_{k=1}^n \mu_k(t) dM_k(t), \end{aligned}$$

hence

$$\sum_{j=1}^n X_j(t) = \mathcal{E} \left(\sum_{k=1}^n \int_0^t \mu_k(s) dM_k(s) \right).$$

Now fix some $i \in \{1, \dots, n\}$ and set

$$\tilde{M}_i(t) := M_i(t) - \sum_{j=1}^n \int_0^t \mu_j(s) dM_j(s) = - \sum_{j=1}^n \int_0^t \tilde{\mu}_j(s) dM_j(s),$$

where

$$\tilde{\mu}_j(t) = \begin{cases} 1 - \mu_i(t) & j = i \\ \mu_j(t) & j \neq i \end{cases}.$$

We get from our Assumption (3), together with $\sqrt{n} \|\mu\| \geq |\mu_1 + \dots + \mu_n| = 1$, that

$$-\kappa t \leq -\frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \int_0^t \mu_j(s) d[M_j, M_k]_s \mu_k(s) \leq -\frac{\varepsilon}{n} t, \quad (4)$$

$$\varepsilon t \leq \frac{1}{2} [M_i]_t \leq \kappa t, \quad (5)$$

and estimate

$$\begin{aligned}
\mu_i(t) &< 1 - \delta \\
\iff \log X_i(t) &< \log(1 - \delta) + \log \sum_{j=1}^n X_j(t) \\
\iff \tilde{M}_i(t) &< a + \frac{1}{2} [M_i]_t - \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \int_0^t \mu_j(s) \, d[M_j, M_k]_s \, \mu_k(s),
\end{aligned}$$

where $a = \log \sum_{j=1}^n X_j(0) - \log X_i(0) + \log(1 - \delta) > 0$ since $\mu_i(0) \leq \mu_{\max}(0) < 1 - \delta$. Therefore, with $b = \kappa - \varepsilon > 0$ and using both the left-hand sides of (4) and (5), we arrive at

$$\mu_i(t) < 1 - \delta \quad \text{if } \tilde{M}_i(t) < a - bt. \quad (6)$$

$\tilde{M}_i$ is a continuous local martingale with $\tilde{M}_i(0) = 0$. We can estimate its quadratic variation by Assumption (3) as

$$\begin{aligned}
[\tilde{M}_i]_t &= \sum_{j=1}^n \sum_{k=1}^n \int_0^t \tilde{\mu}_j(s) \, d[M_j, M_k]_s \, \tilde{\mu}_k(s) \\
&\geq 2\varepsilon \int_0^t \|\tilde{\mu}(s)\|^2 \, ds \\
&\geq 2\varepsilon \int_0^t (1 - \mu_i(s))^2 \, ds
\end{aligned} \quad (7)$$

as well as, using $\|\tilde{\mu}\|^2 \leq |\tilde{\mu}_1 + \dots + \tilde{\mu}_n|^2 = 4(1 - \mu_i)^2$,

$$\begin{aligned}
[\tilde{M}_i]_t &\leq 2\kappa \int_0^t \|\tilde{\mu}(s)\|^2 \, ds \\
&\leq 8\kappa \int_0^t (1 - \mu_i(s))^2 \, ds \\
&\leq 8\kappa t.
\end{aligned} \quad (8)$$

To show that $P^0(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta) < 1$ we use that $\tilde{M}_i$ is a time-changed Brownian motion. Indeed, by Karatzas and Shreve (1991), Theorem 3.4.6 and Problem 3.4.7, there exists on a suitably extended probability space a Brownian motion B with $B_0 = 0$ such that $\tilde{M}_i(t) = B_{[\tilde{M}_i]_t}$ for $t \geq 0$. In particular, by the construction of this extension as carried out in Karatzas and Shreve (1991), Remark 3.4.1, we can take B to be a P -Brownian motion simultaneously for all $P \in \mathcal{M}^e(X)$ (we need this to prove the other inequality in ND). Since $a > 0$ we

can choose $T_i > 0$ small enough such that $a - bT_i \geq \rho$ for some $\rho > 0$ and such that

$$P^0(B_t < \rho \quad \text{for all } t \in [0, 8\kappa T_i]) > 1 - \frac{1}{n}.$$

By using (8) we estimate

$$\begin{aligned} & P^0(\tilde{M}_i(t) < a - bt \quad \text{for all } t \in [0, T_i]) \\ &= P^0(B_{[\tilde{M}_i]_t} < a - bt \quad \text{for all } t \in [0, T_i]) \\ &\geq P^0(B_{[\tilde{M}_i]_t} < \rho \quad \text{for all } t \in [0, T_i]) \\ &= P^0(B_t < \rho \quad \text{for all } t \in [0, [\tilde{M}_i]_{T_i}]) \\ &\geq P^0(B_t < \rho \quad \text{for all } t \in [0, 8\kappa T_i]) > 1 - \frac{1}{n}. \end{aligned}$$

From this and (6) we can conclude by setting $T = \min_{1 \leq i \leq n} T_i$ that for all $i \in \{1, \dots, n\}$

$$P^0\left(\sup_{0 \leq t \leq T} \mu_i(t) < 1 - \delta\right) > 1 - \frac{1}{n},$$

hence

$$P^0\left(\sup_{0 \leq t \leq T} \mu_{\max}(t) < 1 - \delta\right) > 0,$$

which is equivalent to

$$P^0\left(\sup_{0 \leq t \leq T} \mu_{\max}(t) \geq 1 - \delta\right) < 1,$$

as desired.

To show that $\inf_{P \in \mathcal{M}^e(X)} P(\sup_{0 \leq t \leq T} \mu_i(t) \geq 1 - \delta) > 0$ we use the lower bound (7) for $[\tilde{M}_i]$. Moreover, here the time horizon $T > 0$ can be arbitrary. Similarly as before, we use the right-hand sides of (4) and (5) to show (now with $b = \kappa - \varepsilon/n > 0$) that $\tilde{M}_i(t) \geq a + bt$ implies $\mu_i(t) \geq 1 - \delta$. We consider the following two cases.

Case 1

$$\inf_{P \in \mathcal{M}^e(X)} P\left(\int_0^T (1 - \mu_i(s))^2 ds \leq \delta^2 T\right) > 0$$

for some $T > 0$. As

$$\begin{aligned} (1 - \mu_i(t))^2 &\leq \delta^2 \quad \text{for some } t \in [0, T] \\ \iff \mu_i(t) &\geq 1 - \delta \quad \text{for some } t \in [0, T] \end{aligned}$$

it follows that

$$\inf_{P \in \mathcal{M}^e(X)} P(\mu_i(t) \geq 1 - \delta \quad \text{for some } t \in [0, T]) > 0.$$

This is what we wanted to show.

Case 2

$$\inf_{P \in \mathcal{M}^e(X)} P\left(\int_0^T (1 - \mu_i(s))^2 ds \leq \delta^2 T\right) = 0$$

for all $T > 0$. Consider a minimizing sequence $(P_n) \subset \mathcal{M}^e(X)$ such that

$$\begin{aligned} P_n(\tilde{M}_i(t) \geq a + bt \quad \text{for some } t \in [0, T]) \\ \searrow \inf_{P \in \mathcal{M}^e(X)} P(\tilde{M}_i(t) \geq a + bt \quad \text{for some } t \in [0, T]). \end{aligned}$$

We may assume that

$$\lim_{n \rightarrow \infty} P_n\left(\int_0^T (1 - \mu_i(s))^2 ds \leq \delta^2 T\right) = 0 \quad (9)$$

[otherwise we are either in Case 1 (for (P_n)) or can extract a further subsequence fulfilling (9)]. Again we proceed by time-changing the process $\tilde{M}_i$ into a Brownian motion B as above:

$$\begin{aligned} & \inf_{P \in \mathcal{M}^e(X)} P(\tilde{M}_i(t) \geq a + bt \quad \text{for some } t \in [0, T]) \\ &= \lim_{n \rightarrow \infty} P_n(B_{[\tilde{M}_i]_t} \geq a + bt \quad \text{for some } t \in [0, T]) \\ &\geq \lim_{n \rightarrow \infty} P_n(B_{[\tilde{M}_i]_t} \geq a + bT \quad \text{for some } t \in [0, T]) \\ &= \lim_{n \rightarrow \infty} P_n(B_t \geq a + bT \quad \text{for some } t \in [0, [\tilde{M}_i]_T]) \\ &\geq \lim_{n \rightarrow \infty} P_n\left(B_t \geq a + bT \quad \text{for some } t \in \left[0, 2\varepsilon \int_0^T (1 - \mu_i(s))^2 ds\right]\right) \\ &\geq \lim_{n \rightarrow \infty} P_n\left(B_t \geq a + bT \quad \text{for some } t \in [0, 2\varepsilon \delta^2 T], \int_0^T (1 - \mu_i(s))^2 ds > \delta^2 T\right) \\ &> 0, \end{aligned}$$

where the last inequalities follow by (7), (9) and the fact that

$$P_n(B_t \geq a + bT \quad \text{for some } t \in [0, 2\varepsilon \delta^2 T])$$

does not depend on n . This gives us our second result. $\square$

Remark 2.12 We cannot guarantee the existence of an arbitrage opportunity if the number of stocks in the portfolio goes to infinity. In that case we cannot choose by our construction a non-zero T such that ND holds up to T . This justifies a criticism of D. Hobson which is based on the fact that one can often observe that new firms are created once one company enjoys a dominant position in some market.

2.4 On the non-degeneracy condition in the standard Itô model

Here we will show that Condition 3 is satisfied in the standard Itô model as used in Fernholz et al. (2005). This shows in particular that their assumptions imply Assumption ND and hence the existence of arbitrage opportunities after the measure change to Q . The prices of n stocks are modelled by the following linear stochastic differential equation:

$$\frac{dX_i(t)}{X_i(t)} = b_i(t) dt + \sum_{v=1}^m \xi_{iv}(t) dW_v(t), \quad X_i(0) = x_i, \quad t \in [0, \infty),$$

for $i = 1, \dots, n$, where W is a standard m -dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, $m \geq n$. The coefficients are assumed to be adapted and finite; moreover,

$$\int_0^t \|b(s)\|^2 ds < \infty, \quad \forall t \in (0, \infty).$$

Denote $\sigma = \xi \xi'$ where $\xi = (\xi_{iv})_{1 \leq i \leq n, 1 \leq v \leq m}$. We shall need the crucial (but common) assumption that the market is *non-degenerate* (in the terminology of Fernholz 1999), i.e.

$$x' \sigma(t) x \geq \epsilon \|x\|^2, \quad \forall x \in \mathbb{R}^n, \quad \forall t \in [0, \infty), \quad (10)$$

and has *bounded variance*

$$x' \sigma(t) x \leq M \|x\|^2, \quad \forall x \in \mathbb{R}^n, \quad \forall t \in [0, \infty) \quad (11)$$

for some real constants $M > \epsilon > 0$.

Condition (10) allows us to remove the drift terms b_i using Girsanov's theorem so that we get the following dynamics of the price processes:

$$\frac{dX_i(t)}{X_i(t)} = \sum_{v=1}^m \xi_{iv}(t) dB_v(t), \quad i = 1, \dots, n, \quad t \in [0, \infty),$$

where B is a standard m -dimensional Brownian motion under some martingale measure P^0 (see section 5.8 of Karatzas and Shreve 1991).

To apply the previous results, we need to show Condition 3. In our case this amounts to the existence of two numbers $0 < \varepsilon < \kappa$ such that for all $\mathbb{R}^n$ -valued processes η we have

$$\begin{aligned} \varepsilon \int_0^t \|\eta(s)\|^2 ds &\leq \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n \int_0^t \eta_j(s) \sum_{v=1}^m \xi_{jv}(s) \xi_{kv}(s) ds \eta_k(s) \\ &\leq \kappa \int_0^t \|\eta(s)\|^2 ds. \end{aligned}$$

The left inequality follows from the non-degeneracy condition (10) while the right-hand side follows because of the condition of bounded variance (11).

Remark 2.13 It is apparent from the preceding discussion that the validity of Assumption ND, and hence the existence of the arbitrage opportunity, depends crucially on the conditions (10), (11) of non-degeneracy and bounded variance. While it seems to be reasonable from an economic point of view to assume that actual markets are diverse, and that the regulatory impact ensuring diversity corresponds in the mathematical model to the measure change to Q , a stalwart of the efficient market hypothesis might object to (pre-)models where those conditions are fulfilled.

3 The dynamics of the price processes under Q : further applications

This section has been included for the convenience of the reader: we review related results and put them into the context of our setting.

3.1 Q -Dynamics of the price processes

Fernholz et al. (2005) construct an explicit example of price processes which lead to a diverse market. With our approach we can generate a multitude of diverse markets: every pre-model as in Section 2.1 satisfying ND leads to a diverse market when seen under Q as defined in (1). Let us now illustrate the new dynamics under the measure Q . For this, we shall make use of Lenglart's extension of Girsanov's theorem (see, Lenglart 1977).

Theorem 3.1 (Lenglart's theorem) *Let Q be a probability measure absolutely continuous with respect to P^0 . Define the process Z as*

$$Z_t = E_{P^0} \left(\left. \frac{X \left\{ \frac{dQ}{dP^0} > 0 \right\}}{P^0 \left(\frac{dQ}{dP^0} > 0 \right)} \right| \mathcal{F}_t \right).$$

Let X be a continuous local martingale under P^0 . Then there exists an X -integrable process α such that

$$X - \int \frac{1}{Z_-} d[Z, X] = X - \int \alpha d[X]$$

is a Q -local martingale.

Although it seems in general difficult to find an explicit expression for the drift α in our situation, let us recall from Jacod (1979) a more detailed description via the Kunita–Watanabe decomposition. In our case, the price processes are given as

$$\frac{dX(t)}{X(t)} = dM(t), \quad t \geq 0,$$

where M is a continuous local martingale under P^0 . We note that the process Z in Lenglart's theorem is square-integrable. Using the Galtchouk–Kunita–Watanabe decomposition, we project Z on the space of all square-integrable martingales which can be written as $\int \gamma dM$ for some predictable process γ and write the resulting orthogonal projection as $\int \beta dM$. The process $\int \beta dM$ is square-integrable by construction and hence we have a fortiori that

$$\int_0^T \beta'(t) d[M]_t \beta(t) < \infty \quad P^0 - \text{a.s.}$$

Moreover, $[Z, M] = \int \beta d[M]$. Z is Q -a.s. strictly positive (see, Revuz and Yor 2004, Proposition VIII.1.2.), so we may set

$$\alpha = \frac{\beta}{Z_-}.$$

It results that

$$\begin{aligned} \int \frac{1}{Z_-} d[Z, M] &= \int \frac{\beta}{Z_-} d[M] \\ &= \int \alpha d[M] \end{aligned}$$

and

$$\int_0^T \alpha'(t) d[M]_t \alpha(t) = \int_0^T \beta'(t) \frac{1}{Z_{t-}^2} d[M]_t \beta(t) < \infty \quad Q - \text{a.s.} \quad (12)$$

Summing up, the dynamics of X under Q are given as

$$\frac{dX}{X} = d\tilde{M} + \alpha d[\tilde{M}], \quad (13)$$

where $\tilde{M} = M - \int \alpha d[M]$ is a local Q -martingale.

3.2 Valuation of claims when the pre-model is complete

Let us now briefly discuss the problem of valuation of claims in our setting. First we observe that defining a price based on superreplication, using admissible integrands, would lead to a non-finite price in our case. Fortunately, it turns out that pricing is still possible if we only allow strategies which require no intermediate credit. We assume that the market under P^0 is complete and apply the traditional replication approach to find a price for contingent claims in our diverse market. Here we review two approaches taken in the literature and show that they are in fact equivalent.

Gossen-Dombrowsky (1992) proposes to consider for any integrable claim H a modified claim $\tilde{H} = H\chi_{\Omega_Q}$, where Ω_Q is the support of the measure Q . He then assigns to it the usual no-arbitrage price $E_{P^0}[\tilde{H}]$ since P^0 is the unique martingale measure for X under the completeness assumption. This method coincides with the approach taken in Fernholz et al. (2005) which, generalized to our setting, is as follows: motivated by (13), we consider the stochastic exponential

$$L = \mathcal{E}\left(-\int \alpha d\tilde{M}\right).$$

It follows from (12) that L is Q -a.s. strictly positive. The proposed price for the claim H is then $E_Q[HL_T]$ which, as

$$L_T \frac{dQ}{dP^0} = \chi_{\Omega_Q} \quad Q - \text{a.s.},$$

coincides with the price as in Gossen-Dombrowsky (1992). Note that for $H \geq 0$ Q -a.s, we get for the associated value process

$$V. = E_{P^0}[H\chi_{\Omega_Q}|\mathcal{F}_t] \geq 0 \quad Q - \text{a.s.}$$

3.3 Further arbitrage opportunities in stock, bond and currency markets

The original motivation of Gossen-Dombrowsky (1992) comes from a model where the stock price follows a geometric Brownian motion respecting two a priori fixed exponential curves as upper and lower boundaries. Delbaen and Schachermayer (1995) show that there are arbitrage possibilities in Bessel processes. Using a Brownian motion B starting in one they construct a new measure which assigns probability zero to the set of paths where B ever hits zero.

In a similar spirit, one gets arbitrage opportunities in bond and currency markets once certain bounds have been imposed. While mathematically these situations are much more straightforward to treat than the case of diverse markets, we shall still give a brief illustration of an exchange mechanism where the domestic currency is tied to some foreign currency by allowing it to float freely only within a certain range. As usual, we work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^0)$. The exchange rate process X , which is used to convert foreign payoffs into domestic currency, is modelled under P^0 for simplicity by the following stochastic differential equation:

$$\frac{dX(t)}{X(t)} = \sigma dW(t),$$

where σ is some positive constant and W is a P^0 -Brownian motion. We now assume that by regulation, X is restricted to move only in a range of $[a, b]$ for some $b > a > 0$ (whether the regulating authority is able to support the currency in this manner is of no concern to us here, albeit it is of highly practical relevance).

Fix a finite time horizon $T > 0$. Observe that in our set-up (where it is assumed that $X(0) \in (a, b)$) we have

$$P^0(a \leq X(t) \leq b \quad \forall t \in [0, T]) > 0$$

and

$$P^0(\exists t \in [0, T] \text{ such that } X(t) \notin [a, b]) > 0.$$

We now pass over to a new measure Q (reflecting the regulatory impact) which is defined via its density

$$Z = \frac{dQ}{dP^0} \Big|_{\mathcal{F}_T} = \begin{cases} 0 & \text{if } X(t) \notin [a, b] \text{ for some } t \in [0, T] \\ c & \text{otherwise} \end{cases},$$

where c is a normalizing constant. We consider the contingent claim

$$f = \chi_{\{Z > 0\}}.$$

As the original market under P^0 is complete, there exists a replicating strategy which, under Q , represents an arbitrage opportunity since $Q(Z = 0) = 0$. This can be seen by the same argument as in the proof of Proposition 2.8 (here it is even easier since we are in a complete-market situation).

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